Nonlinear phase shift from photon-photon scattering in vacuum

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We show that QED nonlinear effects imply a phase correction to the linear evolution of electromagnetic waves in vacuum. We provide explicit solutions of the modified Maxwell's equations for the propagation of a superposition of two plane waves, and calculate analytically and numerically the corresponding phase shift. This provides a new framework for the search of all-optical signatures of photon-photon scattering in vacuum. In particular, we propose an experiment for measuring the phase shift in projected high-power laser facilities.

PACS numbers: 12.20.Ds, 42.50.Xa, 12.20.Fv

Introduction.- Quantum Electrodynamics (QED) predicts that at photon energies well below the electron rest energy, photon-photon collisions can still be produced through the interchange of virtual electron-positron pairs[1, 2]. This nonlinear interaction modifies Maxwell's equations for the average values of the electromagnetic quantum fields[3] and affects the properties of the QED vacuum[4]. During many years, the search of these effects has been restricted to projected particle physics experiments with accelerators. However, photon-photon scattering processes in vacuum will become testable at energy densities achievable with ultra-high power lasers in the near future[5].

The realization of these ultra intense photon sources began with the discovery of chirped pulse amplification (CPA)[6] in the late 80's and optical parametric chirped pulse amplification (OPCPA) [7] in the 90's. These techniques opened the door to a field of research in the boundary between optics and experimental high-energy physics, where lots of novelties are expected to come in the next years. In fact, several recent works propose different configurations that can be used to test the nonlinear optical response of the vacuum, e.g. using harmonic generation in an inhomogeneous magnetic field[8], QED four-wave mixing[9], resonant interactions in microwave cavities[10], or QED vacuum birefringence[11] which can be probed by x-ray pulses[12], among others[13].

In the present Letter, we show that QED nonlinear effects in vacuum imply a phase correction to the linear evolution of crossing electromagnetic waves. We provide explicit numerical and analytical approximate solutions for the propagation of a superposition of two plane waves. This result allows us to calculate the corresponding phase shift, providing a new framework for the quest of signatures of photon-photon scattering. In particular, we suggest an experiment for measuring the effect of nonlinear vacuum in projected high-power laser facilities like the European Extreme Light Infrastructure (ELI) project[14] for near IR radiation.

Model and equations-. Let us begin by writing the formulae that have to be used instead of the classical linear Maxwell equations, including the terms which come from QED effects in vacuum. The corresponding Lagrangian density in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, was derived in the 30's by Euler and Heisenberg[2]:

$$\mathcal{L} = \mathcal{L}_0 + \xi \mathcal{L}_Q = \mathcal{L}_0 + \xi \left[\mathcal{L}_0^2 + \frac{7\epsilon_0^2 c^2}{4} (\mathbf{E}\cdot\mathbf{B})^2 \right], \quad (1)$$

being

$$\mathcal{L}_0 = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 - c^2\mathbf{B}^2\right) \quad (2)$$

the linear Lagrangian density and ϵ_0 and c the dielectric constant and the speed of light in vacuum, respectively. As it can be appreciated in Eq. (1), QED corrections are introduced by the parameter

$$\xi = \frac{8\alpha^2\hbar^3}{45 m_e^4 c^5} \simeq 6.7\times 10^{-30}\frac{m^3}{J}. \quad (3)$$

This quantity has dimensions of the inverse of an energy density. This means that significant changes with respect to linear propagation can be expected for values around $|\xi\mathcal{L}_0| \sim 1$, corresponding to beam fluxes with electromagnetic energy densities given by the time-time component of the energy-momentum tensor

$$T_{00} = \frac{\partial \mathcal{L}}{\partial(\partial_t A)}\partial_t A - \mathcal{L} \gtrsim 2/\xi \simeq 3\times 10^{29} J/m^3. \quad (4)$$

While such intensities may have an astrophysical or cosmological importance, they are not achievable in the laboratory. The best high-power lasers that are being projected for the next few decades will be several orders of magnitude weaker, eventually reaching energy density of the order $\rho \sim 10^{23} J/m^3$[5]. Therefore, we will study here the "perturbative" regime, in which the non-linear correction is very small, $|\xi\mathcal{L}_0| \ll 1$. As we shall see, even in this case measurable effects can be accumulated in the phase of beams of wavelength λ traveling over a distance of the order $\lambda|\xi\mathcal{L}_0|^{-1}$. Thus, current sensitive techniques could be used to detect traces of QED vacuum nonlinearities.

Once the electromagnetic fields are expressed in terms of the four-component gauge field $A^\mu = (A^0, \mathbf{A})$ as $\mathbf{B} = \nabla \wedge \mathbf{A}$ and $\mathbf{E} = -c\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}$, the equations of motion are given by the Variational Principle:

$$\frac{\delta \Gamma}{\delta A^\mu} = 0, \tag{5}$$

where $\Gamma \equiv \int \mathcal{L} d^4x$ is the QED effective action. Instead of studying the resulting equations for the fields $\mathbf{E}$ and $\mathbf{B}$, that can be found in the literature[3, 15], for the present purposes it is more convenient to consider the equations for the gauge field components A^μ. In general, these four equations cannot be disentangled. However, after some straightforward algebra it can be seen that they admit solutions in the form of linearly polarized waves, e.g. in the x direction, with $A^0 = 0$ and $\mathbf{A} = (A, 0, 0)$, provided that: *i*) the field A does not depend on the variable x (a *transversality* condition) and *ii*) $A(t, y, z)$ satisfies the single equation:

$$\partial_\mu \partial^\mu A + \xi\epsilon_0 \left(\partial_\mu \partial^\mu A \partial_\nu A \partial^\nu A + 2\partial_\mu A \partial^\mu_\nu A \partial^\nu A\right) = 0, \tag{6}$$

where we have used the convention $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ for the metric tensor. Hereafter, we will restrict our discussion to this case of linearly-polarized solution. The orthogonality relation $\mathbf{E} \cdot \mathbf{B} = 0$ are then automatically satisfied, and the effective Lagrangian Eq. (1) reduces to $\mathcal{L} = \mathcal{L}_0 (1 + \xi\mathcal{L}_0)$. Note that the plane-wave solutions of the linear Maxwell equations, such as $\mathcal{A}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$, where $\mathcal{A}$ is a constant, $\mathbf{k} = (0, k_y, k_z)$ and $\omega = c|\mathbf{k}|$, are still solutions of Eq. (6). However, we expect that the non-linear terms proportional to ξ, due to the QED correction, will spoil the superposition principle.

Variational solutions-. Let us consider two plane waves, for simplicity having the same phase at the space-time origin, having wave vectors $(0, q, k)$ and $(0, -q, k)$ respectively, and angular frequency $\omega = \sqrt{k^2 + q^2}$. Any of them, when taken alone, would be a solution of both the linear and non-linear equations. However, the superposition

$$\begin{aligned} A(t, y, z) &= \frac{\mathcal{A}}{2}\left[\cos(kz - \omega t + qy) + \cos(kz - \omega t - qy)\right] \\ &= \mathcal{A}\cos(qy)\cos(kz - \omega t), \end{aligned}$$

where $\mathcal{A}$ is a constant, would only solve the linear equations of motion. In our perturbative regime, we can expect that the small non-linear correction will progressively modify the form of $A(t, y, z)$ as the wave proceeds along the z direction. We will therefore make the ansatz

$$A = \mathcal{A}\cos(qy)\left[\,\alpha(z)\cos(kz - \omega t) + \beta(z)\sin(kz - \omega t)\,\right], \tag{7}$$

allowing for the generation of the other linearly-independent function $\sin(kz - \omega t)$ (we will take $\alpha(0) = 1$ and $\beta(0) = 0$). On the other hand, we neglect the possible generation of a reflected wave depending on $kz + \omega t$, which can be expected to be a smaller correction in this perturbative regime. As we will discuss below, in our perturbative regime the effects of the possible y-dependence of α and β are negligible for the traveling distances in the z direction that we will consider.

According to the Variational Method, we require that the functions $\alpha(z)$ and $\beta(z)$ correspond to a local minimum of the effective action Γ, after averaging out y and t as follows:

$$\Gamma = \int_{-\infty}^{\infty} dz \left(\frac{q\omega}{4\pi^2}\int_{-\pi/q}^{\pi/q} dy \int_0^{2\pi/\omega} dt\mathcal{L}\right), \tag{8}$$

whose minimum corresponds to the equations $\delta\Gamma/\delta\alpha = 0$ y $\delta\Gamma/\delta\beta = 0$. After a straightforward computation, these two equations can be written as

$$\begin{aligned} &\frac{\alpha''}{2} + k\beta' + aq^4\alpha(\beta^2 + \alpha^2) - \frac{a}{16}\left[2\left(9k^2 - 4q^2\right)\beta^2\right. \\ &\quad \left. + 54k\beta\alpha' + 27\alpha'^2 + 6k^2\alpha^2 - 18k\alpha\beta' + 9\beta'^2\right]\alpha'' \\ &+ \frac{a}{8}\left[9k\alpha\alpha' - 9\alpha'\beta' - 9k\beta\beta' + 2\left(3k^2 - 2q^2\right)\alpha\beta\right]\beta'' \\ &+ \frac{a}{8}\left[-9k\beta'^3 - 3k^2\alpha\alpha'^2 + \left(15k^2 - 8q^2\right)\alpha\beta'^2\right] \\ &- \frac{a}{8}\left[9k\alpha'^2 - 16kq^2\alpha^2 - 16kq^2\beta^2 + 2\left(9k^2 - 4q^2\right)\alpha'\beta\right]\beta' \\ &= 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\frac{\beta''}{2} - k\alpha' + aq^4\beta(\alpha^2 + \beta^2) - \frac{a}{16}\left[2\left(9k^2 - 4q^2\right)\alpha^2\right. \\ &\quad \left. - 54k\alpha\beta' + 27\beta'^2 + 6k^2\beta^2 + 18k\beta\alpha' + 9\alpha'^2\right]\beta'' \\ &+ \frac{a}{8}\left[-9k\beta\beta' - 9\beta'\alpha' + 9k\alpha\alpha' + 2\left(3k^2 - 2q^2\right)\beta\alpha\right]\alpha'' \\ &+ \frac{a}{8}\left[9k\alpha'^3 - 3k^2\beta\beta'^2 + \left(15k^2 - 8q^2\right)\beta\alpha'^2\right] \\ &- \frac{a}{8}\left[-9k\beta'^2 + 16kq^2\beta^2 + 16kq^2\alpha^2 + 2\left(9k^2 - 4q^2\right)\beta'\alpha\right]\alpha' \\ &= 0, \end{aligned} \tag{10}$$

where $a = \xi\epsilon_0\mathcal{A}^2/2$. It is now convenient to express the parameter a in terms of the energy density,

$$\begin{aligned} \rho &= T_{00} \\ &= \frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) + \frac{\xi}{4}\epsilon_0^2(\mathbf{E}^2 - c^2\mathbf{B}^2)\left(3\mathbf{E}^2 + c^2\mathbf{B}^2\right). \end{aligned} \tag{11}$$

In our perturbative regime, this gives $a \simeq \frac{2c^2}{\omega^2}\xi\rho$ with a very good approximation.

Numerical simulations and approximate analytical solution-. The result of the numerical integration of Eqs. (9) and (10) is shown in Fig. 1 for a choice of parameters that may be accessible at future facilities[5], namely:

$\rho = 4 \times 10^{23} J/m^3$, $\lambda = 5 \times 10^{-7} m$, $k = q = \frac{2\pi}{\sqrt{2}\lambda} = 0.89 \times 10^7 m^{-1}$, giving $a = 3.4 \times 10^{-20} m^2$. The two enveloping functions show a sinusoidal behavior, with an oscillation length $\sim 0.26m$. In particular, we find that a 100% change in phase is accumulated after the distance $\Delta z \simeq 13cm$.

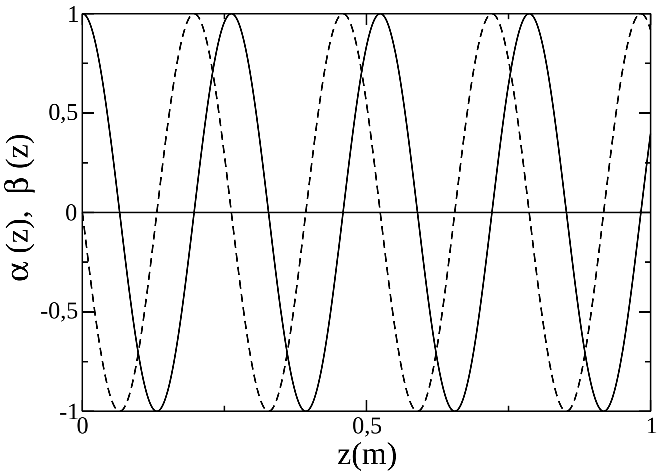

FIG. 1: Numerical solution of Eqs. (9) and (10) for $k = q = 0.89 \times 10^7 m^{-1}$ and $a = 3.4 \times 10^{-20} m^2$. The continuous curve represents the function $\alpha(z)$, the dashed curve the function $\beta(z)$.

This numerical result also suggests the viability of an analytical approximation. In fact, the numerical oscillations show that each order in derivation of α and β corresponds to a suppression factor $\sim 2\pi/0.26m^{-1}$, 6 orders of magnitude smaller than k and q. We can then neglect the second derivatives in Eqs. (9), (10). Moreover, in our perturbative regime a can also be considered an expansion parameter, therefore we can also neglect the first derivatives when they appear multiplied by a. As a result, Eqs. (9), (10) can be approximated as follows:

$$k\beta' + aq^4\alpha(\alpha^2 + \beta^2) = 0, \tag{12}$$

$$-k\alpha' + aq^4\beta(\alpha^2 + \beta^2) = 0, \tag{13}$$

whose analytical solution is

$$\alpha(z) = \cos(\chi z), \quad \text{and} \quad \beta(z) = -\sin(\chi z), \tag{14}$$

where

$$\chi \equiv aq^4/k = \xi\epsilon_0\mathcal{A}^2 q^4/2k \simeq 2\frac{c^2q^4}{k\omega^2}\xi\rho. \tag{15}$$

This result fairly coincides with our numerical solution, giving $\chi = 24m^{-1}$ and $2\pi/\chi = 0.26m$ for the same choice of the parameters as in Fig. 1. This is an *a posteriori* justification for the analytical approximation.

By substituting in Eq. (7) and after some trigonometry, we find that our approximated *analytical* solution of the non-linear equations for the electromagnetic field in the vacuum can be written as

$$A(t, y, z) = \mathcal{A}\cos(qy)\cos(kz + \chi z - \omega t). \tag{16}$$

Therefore, the effect of the non-linearity is to change the phase of the wave, with respect to the linear solution, by a term that increases linearly with the distance. This can also be interpreted as a change in the z component of the wave vector, which becomes $k_z = k + \chi$, so that the dispersion relation is modified to $\omega = c\sqrt{(k_z - \chi)^2 + q^2}$.

To conclude this section, let us discuss now the validity of the approximations that we have made. The Variational Method that we have used is expected to provide very good results whenever the class of test functions, given by Eq. (7) in our case, is a reasonably good choice. Since the latter was motivated by perturbative considerations, the whole approximation will be justified whenever the correction $\chi \ll k$, i.e. whenever $2\frac{c^2q^4}{k^2\omega^2}\xi\rho \ll 1$. This is guaranteed by the dispersion relation and the fact that $\xi\rho \ll 1$ in our regime. It is easy to see that in this case the correction to the energy density dependence on k and q is also very small.

The other approximation that we have made was neglecting the y-dependence of the enveloping functions α and β. For an *a posteriori* test of this hypothesis, we have corrected our solution Eq. (16) allowing for the further envelop functions $\gamma(y, z)$ and $\sigma(y, z)$. We have then used as a new ansatz for the Variational Method the potential $A \equiv \mathcal{A}\cos(qy)(\gamma\cos(k_z z - \omega t) + \sigma\sin(k_z z - \omega t))$, where $k_z = k + \chi = k + aq^4/k$ from Eq. (15).

We have performed a first average over the fast variation in y in the action due to the trigonometric dependence on the product qy, and we have minimized the action. After keeping only the first order in the expansion parameter a, and neglecting the terms involving the second derivatives (which imply a slower variation as discussed above), we have got the equations $\partial_z\sigma + \chi\gamma\left(\gamma^2 + \sigma^2 - 1\right) = 0$, and $\partial_z\gamma - \chi\sigma\left(\gamma^2 + \sigma^2 - 1\right) = 0$. These equations, which do not involve $\partial_y\gamma$ and $\partial_y\sigma$ at the first order, are solved by $\gamma(y, z) = 1$ and $\sigma(y, z) = 0$, therefore we conclude that at this order our previous solution, Eq. (16), is not modified. This justifies the approximation of neglecting the possible y-dependence in the first instance.

Comparison with the optical Kerr effect-. Since it is theoretically known that the vacuum shows birefringence as in the DC Kerr effect[4], it is interesting to compare our result with the optical (AC) Kerr effect that also arises in matter. In fact, let us consider a Kerr medium, characterized by an effective nonlinear refractive index of the form:

$$n = n_0 + n_2 I, \tag{17}$$

where n_0 is the linear refractive index, n_2 the Kerr coefficient and I the irradiance of the beam. In such a medium, for propagation through a distance dz, the wavefront phase will be modified by an amount:

$$d\Phi = \frac{2\pi}{\lambda} n_2 I dz. \tag{18}$$

Since $\rho = I/c$, Eqs. (18) and (15) show that our configuration of two crossing waves in vacuum undergoes the same phase shift as a single plane wave in a Kerr medium having $n_0 = 1$ and a Kerr coefficient $n_2 = 2c\xi q^4/(\omega^2 k^2)$. Choosing e.g. $q = k = \omega/(\sqrt{2}c)$, this Kerr coefficient would be $n_2 = \xi/(\sqrt{2}c) \approx 10^{-38} m^2 W^{-1}$. However, in spite of this analogy, it is important to note that such an n_2 cannot be interpreted as a Kerr coefficient for the vacuum. In fact, strictly speaking the vacuum *does not* show the usual AC Kerr effect. In a Kerr medium, the phase shift (18) is found even for a single plane wave propagating along the z-direction. In vacuum, such a single plane wave, corresponding to $q = 0$ in Eq. (15), would propagate with $\chi = 0$, i.e. without any phase change, exactly as in the linear case. Ultimately, this is due the fact that the cross section for photon-photon scattering vanishes for parallel momenta.

Proposal of an experiment-. Our previous discussion suggests that we might test the non-linear properties of the QED vacuum by using sensitive experiments in which the key point is the ability of measuring small phase changes in a laser beam. In our configuration, a laser pulse is divided in three beams of the same intensity and two of the resulting beams are focused independently. The trajectories of both rays cross at the focus point with an angle θ. As a result, the central part of each distribution has acquired a phase shift $\Delta\Phi$. In a experiment corresponding to the parameters of the ELI project in its first step we have pulses of wavelength $\lambda = 800nm$, intensity $I = 10^{29} Wm^{-2}$ and duration $\tau = 10fs$ which are focused in a spot of diameter $d \approx 10\mu m$. Either from Eq. (15), or equivalently from Eq. (18), this results in a phase shift $\Delta\Phi \approx 10^{-7} rad$, which can be resolved comparing with the third beam which was not exposed to the effects of QED vacuum. Current techniques like spectrally resolved two-beam coupling, which can be applied for ultrashort pulses[16], can be used to this purpose. Comparing to other alternatives like x-ray probing of QED birefringence, our system does not need an extra free electron laser and the power requirements of the system are only one order of magnitude higher. Moreover, the measurement of the ellipticity and the polarization rotation angle in birefringence experiments is not yet possible with current technology. Other techniques like four-wave mixing processes[9] require the crossing of at least three beams, with the corresponding alignment problems and the rest of the requirements are similar to our proposal.

Conclusions.- In conclusion, we have calculated the phase shift arising from propagation of ultra-intense radiation in vacuum, and shown that it could be measured in the first step of ELI facility under construction. We consider that the present work could serve as a starting point for the quest of other nonlinear optical phenomena that may arise in ultra-high power laser beams propagating in vacuum.

Acknowledgments.- We thank Miguel Ángel García-March, Pedro Fernández de Córdoba, Davide Tommasini and Mario Zacarés for useful discussions and help. One of the authors (D. T.) would also like to thank the whole InterTech group at the Universidad Politécnica de Valencia for its warm hospitality. This work was partially supported by contracts PGIDIT04TIC383001PR from Xunta de Galicia, FIS2004-02466, FIS2005-01189 and TIN2006-12890 from the Government of Spain.

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